NONISOTHERMAL INSTABILITY OF HIGH-VELOCITY

ELASTOPLASTIC FLOWS

V. M. Volchkov, A. A. Kozlov, and
P. V. Kuzin

UDC 532.135

An important feature of the high-velocity deformation of solids is the localization of deformation, one of the causes of which may be the nonisothermal instability of plastic flow [1-6]. In connection with the intensive development of high-velocity technology in the treatment of materials, the investigation of the criteria for nonisothermal stability of processes of plastic deformation is of fundamental interest, since in certain cases they determine the optimum technological regimes [5]. The critical values of deformation velocities, above which the effects of thermal instability becomes decisive in the process of deformation of solids, are estimated by semiempirical methods in [1]. The non-boundary-value problem of the criteria for nonisothermal instability is analyzed in [2] for the point of view of flow stability in the so-called coupled formulation. The latter means that the heat-conduction equation is added to the basic equations determining the dynamics of an elastoplastic medium. The problem is solved in [6] in an analogous formulation, but for flow averaged over the spatial coordinate. The solution of the boundary-value problem for one-dimensional flow in this formulation is given in the present paper.

1. Maxwell's model of an viscoelastic medium, which satisfactorily describes the behavior of a material at high deformation velocities [7], is adopted below.

In this case, the equations of one-dimensional motion of the medium are written in the form

$$\rho\frac{\partial u}{\partial t}=\frac{\partial\sigma}{\partial y},\ \frac{1}{G}\frac{\partial\sigma}{\partial t}=\frac{\partial u}{\partial y}-\frac{\sigma}{\mu},\ \rho c\frac{\partial T}{\partial t}=\lambda\frac{\partial^2 T}{\partial y^2}+\frac{\sigma^2}{\mu}\qquad(1.1)$$
$$(\mu=\mu_0\exp[-\beta(T-T_0)]),$$

where $\rho$ is the density of the medium; G is the shear modulus; c and $\lambda$ are the specific heat and thermal conductivity of the medium; u is the flow velocity; $\sigma$ is the stress; T is the temperature; $\mu_0$ and $\beta$ are constants in the Reynolds formula for viscosity.

The system (1.1) is investigated for the boundary conditions

$$\partial T/\partial y=0,\ u=V_0\ \text{at}\ y=h,$$
$$\lambda\partial T/\partial y=\alpha(T-T_0),\ u=0\ \text{at}\ y=0.$$

Here $\alpha$ is the heat-transfer coefficient at the boundary; $V_0$ is the velocity of the upper boundary; $T_0$ is the temperature of the ambient medium.

We introduce the dimensionless variables [6]

$$\tilde u=u/V_0,\ \tilde\sigma=\sigma/GDT_0,\ \Theta=\beta(T-T_0),$$
$$\tilde t=t/t_0,\ \tilde y=y/h,$$

where

$$t_0=c\rho h/\alpha;\ t_1=c\rho/(\beta\mu_0D^2);$$
$$t_2=\mu_0/G;\ t_3=h^2\rho/G,\ D=V_0/h;$$

$t_0$, $t_1$, $t_2$, and $t_3$ are the characteristic times of heat outflow, heat release, elastic relaxation, and propagation of elastic waves.

Volgograd. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 133-138, May-June, 1986. Original article submitted January 18, 1985.

 0021-8944/86/2703-0446$12.50 

In these dimensionless variables, Eqs. (1.1) take the form

$$\frac{\partial\tilde{u}}{\partial\tilde{t}} = A\frac{\partial\tilde{\sigma}}{\partial\tilde{y}}, \quad \frac{\partial\tilde{\sigma}}{\partial\tilde{t}} = \frac{\partial\tilde{u}}{\partial\tilde{y}} - \delta\sigma\exp(\Theta), \qquad (1.2)$$

$$\frac{\partial\Theta}{\partial\tilde{t}} = \frac{1}{\mathrm{Bi}}\frac{\partial^2\Theta}{\partial\tilde{y}^2} + \varkappa\delta^2\sigma^2\exp(\Theta)$$

with the boundary conditions

$$\partial\Theta(1, \tilde{t})/\partial\tilde{y} = 0, \quad \partial\Theta(0, \tilde{t})/\partial\tilde{y} = \mathrm{Bi}\,\Theta(0, \tilde{t}), \qquad (1.3)$$

$$\tilde{u}(0, \tilde{t}) = 0, \quad \tilde{u}(1, \tilde{t}) = 1$$

$$(\delta = t_0/t_2, \ \varkappa = t_0/t_1, \ A = t_0/t_3, \ \mathrm{Bi} = \alpha h/\lambda).$$

The tilde, denoting a dimensionless variable, is omitted below. The steady-state solution of the problem (1.2), (1.3) is

$$u_0 = \frac{2c_1}{\delta\varkappa\sigma_0\,\mathrm{Bi}}\left[\frac{1}{1+\exp(-c_1)} - \frac{1}{1+\exp(c_1(y-1))}\right], \qquad (1.4)$$

$$\sigma_0 = \frac{c_1}{\delta\varkappa\,\mathrm{Bi}}\,\frac{1-\exp(-c_1)}{1+\exp(-c_1)} = \mathrm{const},$$

$$\exp(\Theta_0) = \frac{2c_1^2\exp(c_1(y-1))}{\delta^2\varkappa\,\mathrm{Bi}\,\sigma_0^2[1+\exp(c_1(y-1))]^2},$$

where the integration constant $c_1$ is found from the condition

$$\varkappa = \frac{(\exp(c_1)-1)^2}{2\,\mathrm{Bi}\exp(c_1)}\exp\left[\frac{c_1}{\mathrm{Bi}}\left(\frac{\exp(c_1)-1}{\exp(c_1)+1}\right)\right].$$

Setting

$$u = u_0(y) + u'(y)e^{\beta t}, \quad \sigma = \sigma_0(y) + \sigma'(y)e^{\beta t},$$

$$\Theta = \Theta_0(y) + \Theta'(y)e^{\beta t},$$

we obtain the problem of the stability of the steady-state solution (1.4) against small disturbances $u'(y)$, $\sigma'(y)$, $\theta'(y)$:

$$\beta u' = A\,d\sigma'/dy, \qquad (1.5)$$

$$\beta\sigma' = du'/dy - \delta\sigma'\exp(\Theta_0) - \delta\sigma_0\Theta'\exp(\Theta_0),$$

$$\beta\Theta' = \frac{1}{\mathrm{Bi}}\frac{d^2\Theta'}{dy^2} + \varkappa\delta^2\sigma_0^2\Theta'\exp(\Theta_0) + 2\varkappa\delta^2\sigma_0\sigma'\exp(\Theta_0).$$

Here the parameter $\beta$ characterizes the intensity of growth of the disturbances. If $\mathrm{Re}\,\beta > 0$, the flow is unstable. In place of (1.3), we find the boundary conditions for $u'$, $\sigma'$, and $\theta'$:

$$d\Theta'/dy = 0, \ u' = 0 \ \text{at} \ y = 1, \qquad (1.6)$$

$$d\Theta'/dy = \mathrm{Bi}\Theta', \ u' = 1 \ \text{at} \ y = 0.$$

To seek the eigenvalues $\beta$, Eqs. (1.5) and (1.6) were represented in the form of a system of eight ordinary, first-order differential equations (for the real and imaginary parts of the disturbances). Two linearly independent particular solutions, for which the conditions (1.6) are satisfied at the initial point of integration (at $y = 0$), were constructed numerically by the Runge–Kutta method. Then a linear combination of these particular solutions was constructed. The necessity of satisfying the boundary conditions at $y = 1$ leads to a characteristic equation, from which the eigenvalues of the problem are found. This procedure for solving stability problems using the numerical construction of particular solutions by the Runge–Kutta method was discussed in [8].

The results of calculations of the region of instability for different values of the parameters $\delta$, $\kappa$, Bi, and A are presented in Fig. 1 (Bi = 16, 4, 1, and 0.01 for lines 1-4). The flow is unstable in the region adjacent to the $\kappa$ axis. The boundaries for A = 0 with the corresponding values of Bi are shown by dashed lines, while the boundaries for A = 1 are shown by solid lines (the curves for larger A do not vary to within the accuracy of plotting of the graph). It is seen that the boundary of the region depends fundamentally on Bi, the parameter A affects its position to a lesser degree, and the dependence on A is insignificant for Bi < 1.

2. In connection with the indicated fact of the insignificant dependence of the stability boundary on A in the indicated region of values of Bi and $\kappa$, it is advisable to analyze the problem for A = 0 [without the first equation of (1.5)].

In this case, the system of equations for the disturbances has the form

$$\beta\sigma' = -\delta\sigma' \exp(\Theta_0) - \delta\sigma_0\Theta' \exp(\Theta_0), \tag{2.1}$$
$$\beta\Theta' = \frac{1}{\mathrm{Bi}}\frac{d^2\Theta}{dy^2} + \varkappa\delta^2\sigma_0^2\Theta' \exp(\Theta_0) + 2\varkappa\delta^2\sigma_0\sigma' \exp(\Theta_0)$$

with the boundary conditions (1.6).

The solution of the problem was sought by the method described in Sec. 1 and by Galerkin's method (a numerical realization of the method using the QR algorithm [9] for determining eigenvalues was executed by G. A. Korolev) in the form of an expansion

$$\begin{pmatrix}\sigma\\ \Theta\end{pmatrix} = \sum_{n=1}^{\infty}\begin{pmatrix}\alpha_n & \varphi_n\\ \beta_n & \psi_n\end{pmatrix}.$$

As the system of base functions, we chose the eigenfunctions of the problem (1.6), (2.1) for $\theta_0$ = const,

$$\varphi_n = 1/\sqrt{2}\cos\pi n y,$$
$$\psi_n = 1/\|\Phi_n\|(\sin\omega_\Theta y + (\omega_\Theta/\mathrm{Bi})\cos\omega_\Theta y), \tag{2.2}$$

where $\omega_\theta$ is determined from the relation

$$\omega_\Theta \operatorname{tg}\omega_\Theta = \mathrm{Bi}; \tag{2.3}$$

$\|\Phi_n\|$ is a normalization factor. The system of functions (2.2) is complete and orthonormalized in the segment [0; 1].

The results obtained by both methods coincide with those found from the solution of the complete problem (as A → 0), marked by dashed lines in Fig. 1.

For comparison, let us consider the same problem (A = 0), but for a constant value of the steady-state solution $\bar\theta_0$ = const [for $\bar\theta_0$ we can take the expression $\theta_0(y)$ from the third equation of (1.4), averaged over the coordinate $\int_0^1 \Theta_0(y)dy$]. In this case, the problem admits of an analytic solution.

Equations (1.5) are reduced to one equation,

$$d^2\Theta'/dy^2 + M\Theta' = 0,$$

where $M = \mathrm{Bi}\left(\varkappa\delta^2\sigma_0^2\exp(\bar\Theta_0) - \frac{2\varkappa\sigma_0^2\delta^3\exp(2\bar\Theta_0)}{\beta - \delta\exp(\bar\Theta_0)} - \beta\right)$. It is easy to show that the eigenfunctions of the problem (2.3) with the boundary conditions (1.6) will be the functions $\psi_n$ from (2.2) in which $\omega_\theta^2$ = M.

For unstable disturbances, Re $\beta$ > 0 (for at least one root). Therefore, using the Hurwitz conditions, from the characteristic equation we find the criterion for flow instability:

$$\omega_\Theta^2/\mathrm{Bi} + \delta\exp(\bar\Theta_0) - \varkappa\delta^2\sigma_0^2\exp(\bar\Theta_0) < 0. \tag{2.4}$$

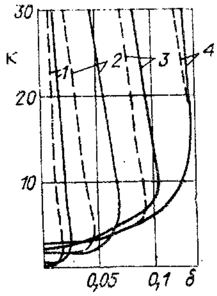

Fig. 1

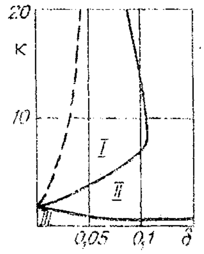

Fig. 2

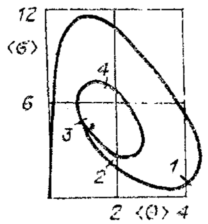

Fig. 3

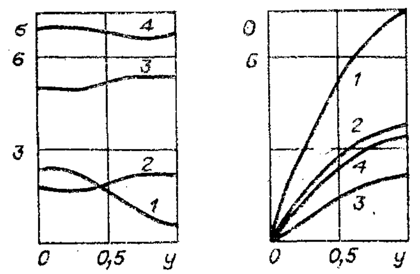

Fig. 4

A = 0.1, Bi = 4, $\varkappa$ = 4, δ = 0.05

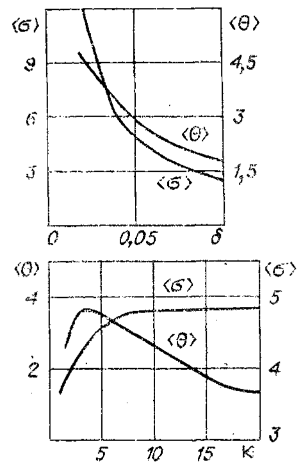

Fig. 5

The boundaries of the regions defined by this relation coincide, to within the accuracy in plotting the graph for the different Bi, with the dashed curves in Fig. 1, obtained from the solution of the complete problem.

As Bi → 0, we have $\omega_\theta^2/\mathrm{Bi} \to 1$ from (2.3), and (2.4) changes into the condition

$$1+\delta\exp(\bar{\Theta}_0)-\varkappa\delta^2\sigma_0^2\exp(\bar{\Theta}_0)<0,$$

coinciding with the instability criterion of [6].

3. To clarify the character of the nonsteady motion of the medium after stability loss, we made a numerical analysis of the complete nonlinear system (1.2) with the boundary conditions (1.3). The difference scheme was constructed on the basis of the method of integral relations [10].

On a grid $\omega_{h,\tau} = \omega_h \times \omega_\tau$,

$$\omega_h = \{y_i = ih,\ i = \overline{0,\ N},\ hN = 1\},$$
$$\omega_\tau = \{t_j = j\tau,\ j = 0,\ 1,\ 2,\ldots\},$$

the system of equations (1.2) with the conditions (1.3) is approximated on the difference scheme

$$h[^{(\alpha)}(\breve{\sigma}_t + \delta\sigma \exp(\breve{\Theta}) - {}^{(\alpha)}(\sigma_t + \hat{\sigma}\delta \exp \Theta))] + A\tau(\sigma_y^{(\beta)} - \sigma_{\bar{y}}^{(\beta)}) = 0,$$

$$h(^{(\nu)}\Theta - {}^{(\nu)}\hat{\Theta}) + (\tau/\mathrm{Bi})(\Theta_y^{(\varepsilon)} - \Theta_{\bar{y}}^{(\varepsilon)}) - \varkappa\delta^2 h\tau^{(\mu)}(\sigma^2 \exp(\breve{\Theta}))^{(\nu)} = 0,$$

in which the following notation [10] is used:

$$\sigma(y_i, t_j) = \sigma, \ \sigma(y_i \pm h, t_j) = \sigma(\pm 1),$$
$$\sigma(y_i, t_j + \tau) = \hat{\sigma}, \ \sigma(y_i, t_j - \tau) = \breve{\sigma},$$
$$\sigma_y = (\sigma(+1) - \sigma)/h, \ \sigma_{\bar{y}} = (\sigma - \sigma(-1))/h, \ \sigma_t = (\hat{\sigma} - \sigma)/\tau,$$
$$\sigma^{(\alpha)} = \alpha\hat{\sigma} + (1-\alpha)\sigma, \ {}^{(\alpha)}\sigma = \alpha\sigma(+1) + (1-\alpha)\sigma.$$

To estimate the accuracy, we compare the results of calculations with time steps $\tau$ and $\tau/2$. The weight factors of the difference scheme were chosen from the conditions of stability fo the trial-run method [10] used to solve the system of algebraic equations, and they were all taken as 0.5. In choosing the time step in the first approximation, were oriented to relations valid for an ordinary explicit difference scheme:

$$\tau < \frac{1}{\delta} \exp\left(-\max_{0 \leq i \leq N} \Theta_i^j\right), \ \tau < \mathrm{Bi}\, h^2/2. \qquad (3.1)$$

These relations were subsequently refined empirically in the calculation process. In this case the stability of the scheme was assured, as a role, with time steps larger than follows from the relations (3.1).

4. In Fig. 2 we give the characteristic regions of behavior of the solutions for A = 1 and Bi = 1: I is the region of fluctuations about the stationary point (unstable regime). Here the region of stability from the point of view of the linear problem of Secs. 1 and 2 is divided into two (II and III). In region II the stationary point is a stable "focus" while in region III it is a stable "node"; in the first case

$$\lim_{t \to \infty} (f_i(y_0, t) - f_i^{(0)}(y_0)) = 0,$$

while in the second case

$$f_i(y_0, t) - f_i^{(0)}(y_0) = 0, \ t > t^*,$$

where $f_i$ is u, σ, or θ; $f_i^{(0)}$ is the steady-state solution (1.4); $0 \leq y_0 \leq 1$.

In Fig. 3 we present a phase diagram ($\sigma^j$, θ) for the region of self-oscilaltions of the averaged σ and θ for A = 1, Bi = 16, δ = 0.02, and κ = 3.5. The averaging was carried out through the formula $\langle\sigma^j\rangle = \frac{1}{N+1}\sum_{i=0}^{N}\sigma_i^j$. The limiting cycle in this diagram is formed around $\langle\sigma_0\rangle$, $\langle\theta_0\rangle$, the averaged stationary point.

The variation of the distribution of ⟨σ⟩ and ⟨θ⟩ over y as a function of time is shown in Fig. 4. Curves 1-4 in Fig. 4 correspond to points 1-4 in Fig. 3. The function $u(y, t_0)$ at t > 0 differs slightly from u(y, 0) = y.

For the period of the oscillations of ⟨σ⟩ and ⟨θ⟩ (in region I) obtained through the present calculation we can use the same formula as in [6], derived in the solution of the averaged problem, $T = \delta^{-0,71}(2.6 \cdot \varkappa^{-0,5} + 0,21)$, with the same accuracy (10-15%). Thus, in the region of $\kappa \lesssim 20$ under consideration, there is no significant dependence of T on Bi and A.

For the amplitude of the oscillations we present characteristic graphs of the dependence on δ and κ in Fig. 5. Nor is a significant dependence of the amplitudes on Bi and A found for $\kappa \lesssim 20$. It should be noted that a stable calculation by the difference scheme could not be achieved for all values of δ and κ. The approximate boundary of the region of stability of the scheme for Bi = 1 and A = 1 is marked by a dashed line in Fig. 2. For $\kappa > e$, the amplitude of the oscillations of $\langle\sigma\rangle$ and $\langle\theta\rangle$ grows faster than exponentially as $\delta$ decreases (Fig. 5).

The analysis of the linear (with respect to small disturbances) boundary-value problem presented in Secs. 1 and 2 makes it possible to determine the region of thermal instability of the flow of a viscoelastic medium as a function of the dimensionless parameters A, Bi, $\kappa$, and $\delta$. It is shown that for low values of $\kappa$ and Bi, the dependence on the parameter A is insignificant, and the approximate criterion (2.4), obtained analytically, can be used. As Bi $\to$ 0, the latter changes into the criterion found in [6] in an analysis of the averaged (over the spatial coordinate) problem. The numerical solution of the nonlinear problem basically confirmed the results of the linear analysis, and made it possible to establish the laws of development of the flow after the loss of stability.

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